Math 2010 Week 13

Thm (Implicit Function Theorem)

Let $\Omega \subseteq \mathbb{R}^{n+k}$ be open, $F: \Omega \longrightarrow \mathbb{R}^k$ be C^1,

Denote $x = (x_1, \cdots, x_n)$, $y = (y_1, \cdots, y_k)$

$$F(x,y) = \begin{bmatrix} F_1(x,y) \\ \vdots \\ F_k(x,y) \end{bmatrix} = \begin{bmatrix} F_1(x_1, \cdots x_n, y_1, \cdots, y_k) \\ \vdots \\ F_k(x_1, \cdots x_n, y_1, \cdots, y_k) \end{bmatrix}$$

Suppose $(a,b) \in \Omega$, where $a \in \mathbb{R}^n$, $b \in \mathbb{R}^k$, such that

$$F(a,b) = c = \begin{bmatrix} c_1 \\ \vdots \\ c_k \end{bmatrix} \in \mathbb{R}^k$$

and the $k \times k$ matrix

$$\left[\frac{\partial F_i}{\partial y_j}(a,b)\right]_{1 \le i,j \le k} = \begin{bmatrix} \frac{\partial F_1}{\partial y_1}(a,b) & \cdots & \frac{\partial F_1}{\partial y_k}(a,b) \\ \vdots & & \vdots \\ \frac{\partial F_k}{\partial y_1}(a,b) & \cdots & \frac{\partial F_k}{\partial y_k}(a,b) \end{bmatrix} \text{ is invertible}$$

Then $\exists$ open sets $U \subseteq \mathbb{R}^n$ containing a

$V \subseteq \mathbb{R}^k$ containing b and C^1 function

$$\begin{array}{rccc} \varphi: & U & \longrightarrow & V \\ & \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ & x = (x_1, \cdots, x_n) & \longmapsto & y = (y_1, \cdots, y_k) \end{array}$$

such that

① $\varphi(a) = b$

② $F(x, \varphi(x)) = c$

③ $$\underbrace{\left[\frac{\partial \varphi_j}{\partial x_\ell}(a)\right]}_{\substack{k \times n \\ 1 \le j \le k \;\; 1 \le \ell \le n}} = - \underbrace{\left[\frac{\partial F_i}{\partial y_j}(a,b)\right]^{-1}}_{k \times k} \underbrace{\left[\frac{\partial F_i}{\partial x_\ell}(a,b)\right]}_{\substack{k \times n \\ 1 \le i \le k \;\; 1 \le \ell \le n}}$$

Special case A : $k=1$

$F: \Omega \subseteq \mathbb{R}^{n+1} \longrightarrow \mathbb{R}^1$

$F(x,y) = F(x_1, \cdots x_n, y)$

i.e. one constraint $F(x_1,\cdots,x_n,y) = c$

Suppose $a=(a_1,\cdots,a_n) \in \mathbb{R}^n$, $b \in \mathbb{R}$

$F(a,b) = F(a_1,\cdots,a_n,b) = c$

IFT means

If $\frac{\partial F}{\partial y}(a_1,\cdots,a_n,b) \neq 0$, then

$\exists\, y = y(x_1,\cdots,x_n)$ near $(a_1,\cdots,a_n)$

solving the constraint (locally)

$$\begin{cases} F(x_1,\cdots,x_n,y) = c \\ y(a_1,\cdots,a_n) = b \end{cases}$$

In eg2 of last week

eg2 $x^2+y^2+z^2=2$. Solve for $z=z(x,y)$?

$(x,y,z) \longleftrightarrow (x_1,x_2,y)$

$\mathbb{R}^3$ Notation	General Notation
$g(x,y,z) = x^2+y^2+z^2 = 2$	$F(x_1,x_2,y) = x_1^2+x_2^2+y^2 = c$ $a=(a_1,a_2)=(0,1)$ $b=1$ $c=2$
$\frac{\partial g}{\partial z}(0,1,1) = 2 \neq 0$ By IFT, $\exists\, z=z(x,y)$ near $(0,1,1)$ such that $\begin{cases} g(x,y,z(x,y)) = 2 \\ z(0,1) = 1 \end{cases}$	$\frac{\partial F}{\partial y}(a_1,a_2,b) = 2 \neq 0$ By IFT, $\exists\, y=y(x_1,x_2)$ near (a_1,a_2,b) such that $\begin{cases} F(x_1,x_2,y(x_1,x_2)) = c \\ y(a_1,a_2) = 1 \end{cases}$

Rmk $\frac{\partial z}{\partial x}$, $\frac{\partial z}{\partial y}$ can be computed by implicit differentiation

Special case B : $n=1,\ k=2$

$F: \Omega \subseteq \mathbb{R}^{1+2} \longrightarrow \mathbb{R}^2$ Two constraints

$$F(x,y_1,y_2) = \begin{bmatrix} F_1(x,y_1,y_2) \\ F_2(x,y_1,y_2) \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

Suppose (a,b_1,b_2) satisfies the constraints.

then IFT means

If $\begin{bmatrix} \frac{\partial F_1}{\partial y_1}(a,b_1,b_2) & \frac{\partial F_1}{\partial y_2}(a,b_1,b_2) \\ \frac{\partial F_2}{\partial y_1}(a,b_1,b_2) & \frac{\partial F_2}{\partial y_2}(a,b_1,b_2) \end{bmatrix}$ is invertible

then $\exists\, y_1 = y_1(x),\ y_2 = y_2(x)$ near a

solving the constraints (locally)

$$\begin{cases} F_i(x, y_1(x), y_2(x)) = c_i \\ y_i(a) = b_i \end{cases}$$

for $i=1,2$

In eg3 of last week

<u>eg3</u> $\begin{cases} x^2+y^2+z^2 = 2 \\ x + z = 1 \end{cases}$ Solve for $y=y(x),\ z=z(x)$?

$(x,y,z) \longleftrightarrow (x_1, y_1, y_2)$

$\mathbb{R}^3$ Notation	General Notation
$F_1(x,y,z) = x^2+y^2+z^2 = 2$ $F_2(x,y,z) = x + z = 1$	$F_1(x,y_1,y_2) = x^2+y_1^2+y_2^2 = c_1$ $F_2(x,y_1,y_2) = x + y_2 = c_2$ $a=0,\ b=(b_1,b_2)=(1,1),\ c=(c_1,c_2)=(2,1)$
$\begin{bmatrix} \frac{\partial F_1}{\partial y} & \frac{\partial F_1}{\partial z} \\ \frac{\partial F_2}{\partial y} & \frac{\partial F_2}{\partial z} \end{bmatrix}_{(0,1,1)} = \begin{bmatrix} 2 & 2 \\ 0 & 1 \end{bmatrix}$ invertible By IFT, $\exists\, y=y(x),\ z=z(x)$ near $(0,1,1)$ such that $\begin{cases} F_1(x,y(x),z(x)) = 2 \\ F_2(x,y(x),z(x)) = 1 \\ y(0)=1 \\ z(0)=1 \end{cases}$	$\begin{bmatrix} \frac{\partial F_1}{\partial y_1} & \frac{\partial F_1}{\partial y_2} \\ \frac{\partial F_2}{\partial y_1} & \frac{\partial F_2}{\partial y_2} \end{bmatrix}_{(a,b_1,b_2)} = \begin{bmatrix} 2 & 2 \\ 0 & 1 \end{bmatrix}$ invertible By IFT, $\exists\, y_1=y_1(x),\ y_2=y_2(x)$ near (a,b_1,b_2) such that $\begin{cases} F_i(x,y_1(x),y_2(x)) = c_i \\ y_i(a)=b_i \end{cases}$ for $i=1,2$

Rmk We write $F = F(x,y)$ and solve y as a function of x, but in fact, the ordering of the variables is not important.

eg Consider the constraints

$$\begin{cases} xz + \sin(yz - x^2) = 8 \\ x + 4y + 3z = 18 \end{cases}$$

Near $(2,1,4)$, can we solve 2 of the variables as functions of the remaining variable?

Sol

Let $F(x,y,z) = \begin{bmatrix} F_1(x,y,z) \\ F_2(x,y,z) \end{bmatrix}$

$$= \begin{bmatrix} xz + \sin(yz - x^2) \\ x + 4y + 3z \end{bmatrix}$$

$$\nabla F = \begin{bmatrix} z - 2x\cos(yz-x^2) & z\cos(yz-x^2) & x + y\cos(yz-x^2) \\ 1 & 4 & 3 \end{bmatrix}$$

$$\nabla F(2,1,4) = \begin{bmatrix} 0 & 4 & 3 \\ 1 & 4 & 3 \end{bmatrix}$$

$\begin{vmatrix} 0 & 4 \\ 1 & 4 \end{vmatrix} = -4 \neq 0 \xRightarrow{\text{IFT}}$ x, y can be expressed as functions of z near $(2,1,4)$

$\begin{vmatrix} 0 & 3 \\ 1 & 3 \end{vmatrix} = -3 \neq 0 \xRightarrow{\text{IFT}}$ x, z can be expressed as functions of y near $(2,1,4)$

$\begin{vmatrix} 4 & 3 \\ 4 & 3 \end{vmatrix} = 0$ No conclusion on whether y, z can be expressed as functions of x near $(2,1,4)$ from IFT

Rmk y, z are not differentiable of x near $(2,1,4)$.

Otherwise, by implicit differentiation,

$$\begin{bmatrix} 4 & 3 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} \frac{dy}{dx} \\ \frac{dz}{dx} \end{bmatrix} = -\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$ no solution $\Rightarrow$ contradiction

Rmk Implicit Function Theorem has many important applications, such as rigorous proofs of

1. Implicit differentiation
2. Tangent plane of surface $F(x,y,z)=C$
3. Lagrange Multipliers

The following Inverse Function Theorem is equivalent to Implicit Function Theorem

Thm (Inverse Function Theorem)

Let $\Omega \subseteq \mathbb{R}^n$ be open,

$f: \Omega \longrightarrow \mathbb{R}^n$ be C^1, $f(a)=b$

Suppose $Df(a)$ is an invertible $n \times n$ matrix.

Then $\exists$ open sets $U_1, U_2 \subseteq \mathbb{R}^n$, $a \in U_1$, $b \in U_2$ and a C^1 function $g: U_2 \to U_1$ such that

1. $g(b)=a$
2. $g(f(x))=x \quad \forall x \in U_1$

   $f(g(y))=y \quad \forall y \in U_2$

   (g is a local inverse of f : $g=(f|_{U_1})^{-1}$)
3. $Dg(b)=Df(a)^{-1}$

Pf Math 3060

eg $f: \mathbb{R}^2 \to \mathbb{R}^2$

$$f(x,y) = (x^2 - y^2, 2xy)$$

Clearly $f(-x,-y) = f(x,y)$

$\Rightarrow$ f is not injective and has no global inverse

How about local inverse ?

Soln
$$f(x,y) = \begin{bmatrix} x^2 - y^2 \\ 2xy \end{bmatrix}$$

$$Df(x,y) = \begin{bmatrix} 2x & -2y \\ 2y & 2x \end{bmatrix}$$

$$\det Df(x,y) = 4(x^2+y^2) > 0 \Leftrightarrow (x,y) \neq (0,0)$$

By Inverse Function Theorem f is locally invertible with differentiable local inverse

For instance, let $g(u,v)$ be a local inverse of $f(x,y)$ near $(x,y) = (1,-1)$

Then $f(1,-1) = (0,-2) \Rightarrow g(0,-2) = (1,-1)$

$$Dg(0,-2) = Df(1,-1)^{-1} = \begin{bmatrix} 2 & 2 \\ -2 & 2 \end{bmatrix}^{-1} = \frac{1}{4}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

In fact, we can find $g(u,v)$ explicitly.

$$\begin{cases} u = x^2 - y^2 \\ v = 2xy \end{cases}$$

Near $(x,y) = (1,-1)$, $x \neq 0 \Rightarrow y = \frac{v}{2x}$

$$\therefore \quad u = x^2 - \left(\frac{v}{2x}\right)^2$$

$$4x^4 - 4ux^2 - v^2 = 0$$

$$x^2 = \frac{4u \pm \sqrt{(-4u)^2 - 4(4)(-v^2)}}{8}$$

$$= \frac{u \pm \sqrt{u^2+v^2}}{2}$$

Put $(x,y)=(1,-1)$, $(u,v)=(0,-2)$

$$\Rightarrow \quad 1^2 = \frac{0 \pm \sqrt{4}}{2}$$

$$\Rightarrow \text{“}-\text{” is rejected}, \quad x^2 = \frac{u+\sqrt{u^2+v^2}}{2}$$

$$\Rightarrow x = \sqrt{\frac{u+\sqrt{u^2+v^2}}{2}}$$

$$y = \frac{2v}{x} = \frac{2\sqrt{2}\,v}{\sqrt{u+\sqrt{u^2+v^2}}}$$

Hence,

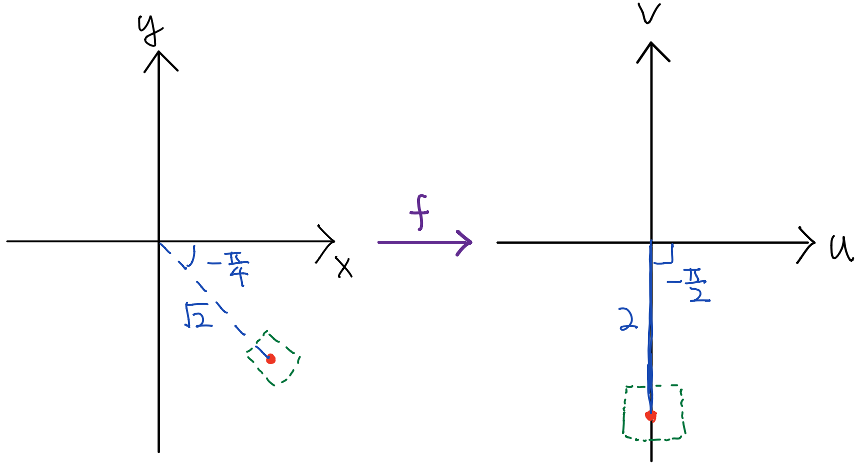
$$g(u,v) = \left(\sqrt{\frac{u+\sqrt{u^2+v^2}}{2}}, \frac{2\sqrt{2}\,v}{\sqrt{u+\sqrt{u^2+v^2}}} \right)$$

Rmk If $z = x + yi \in \mathbb{C}$, then

$$z^2 = x^2 - y^2 + 2xyi$$

$$\therefore \quad u = x^2 - y^2 = \operatorname{Re} z^2$$

$$v = 2xy = \operatorname{Im} z^2$$



$$z \longmapsto z^2$$

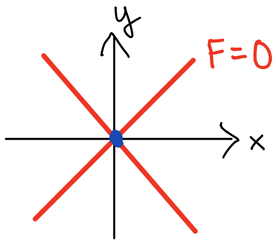
In both Implicit and Inverse Function Theorems, we assume a Jacobi matrix to be invertible.

Without this assumption, the theorems are inconclusive on the existence of local implicit or inverse function. See examples below:

Implicit Function Theorem

$F(x,y) = x^2 - y^2 = 0$

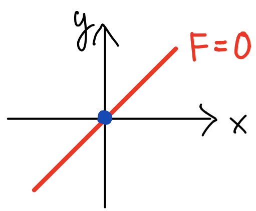
$\left.\frac{\partial F}{\partial y}\right|_{(0,0)} = 0$



y is not locally a function of x near $(0,0)$

$F(x,y) = x^3 - y^3 = 0$

$\left.\frac{\partial F}{\partial y}\right|_{(0,0)} = 0$

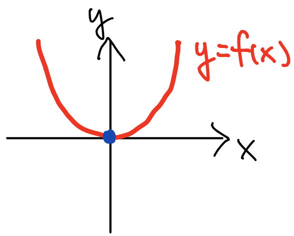


$y = x$ both globally and locally near $(0,0)$

Inverse Function Theorem

$f(x) = x^2$

$f'(0) = 0$

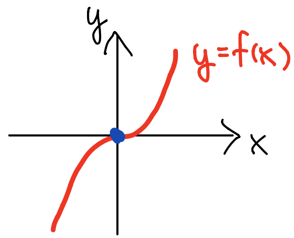


Not injective near $x=0$

$\Rightarrow$ no local inverse near $x=0$

$f(x) = x^3$

$f'(0) = 0$



$g(y) = \sqrt[3]{y}$ is both global and local inverse near $x=0$